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Buckling of a conical thin shell with variable thickness under a dynamic loading

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Abstract

This study considers the buckling of an elastic truncated conical shell having a meridional thickness expressed by an arbitrary function, subject to a uniform external pressure, which is a power function of time. At first, the fundamental relations and Donnell type dynamic buckling equation of an elastic conical shell with variable thickness have been obtained. Then, employing Galerkin's method, those equations have been reduced to a time-dependent differential equation with variable coefficients. Finally, applying the Ritz type variational method, the critical static and dynamic loads, the corresponding wave numbers, dynamic factor and critical stress impulse have been found analytically. Using the results, thus obtained, the effects of the thickness variations with a power or an exponential function, the variation of the semi-vertex angle and the variation of the power of time in the external pressure expression are studied through pertinent computations. It is observed that these factors have appreciable effects on the critical parameters of the problem in the heading.

1. Introduction

Conical shells, including circular cylindrical shells and annular plates as special cases, play an important role in many industrial fields. There are numerous methods to find the values of static critical loads, consistent with experimental results, for shells with constant thickness under different boundary conditions. On the other hand, there are fewer research works about shells with variable thickness, due to the difficulties in production and theoretical analysis. Nonetheless, it is highly probable that these types of structural parts will be used a lot in the future due to the

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advantages of their low weights and small dimensions and the progress in their fabrication methods. In recent years, numerous research works have been published, concerning the buckling and vibration of shells with variable thickness. Bergman et al. [1] derived the equations of free vibration of a hinged cylindrical non-circular shell of variable thickness. Irie et al. [2] and Takahashi et al. [3] have analyzed the free vibration of a truncated conical shell having a meridional thickness expressed by an arbitrary function. Ohga et al. [4,5] presented an analytical procedure for the elastic buckling problems of thin-walled members and curved panels with variable thickness, using the transfer matrix approach. Medvedev [6] analyzed orthotropic non-circular shells with the objective of maximizing the lowest natural frequencies by varying the thickness with a constraint on mass. Using a semi-analytical finite element method, Sankaranarayanan et al. [7] and, Sivadas and Ganesan [8] have studied the effects of thickness variation on the natural frequencies of laminated conical shells. Koiter et al. [9] have studied in detail the buckling of the cylindrical shell with small thickness variations. Results from the asymptotic formulas are compared with those obtained through the purely numerical techniques of the finite difference method and the shooting method. Kang and Leissa [10] presented a three-dimensional method of analysis for determining the natural frequencies and mode shapes of hollow cones and cylinders. Bambil et al. [11] analyzed the transverse vibrations of an orthotropic rectangular plate, with linearly varying thickness. Babic [12] analyzed the natural vibrations of a conical orthotropic shell with small curvatures of the generatrix. Liew et al. have studied the vibration behaviour of shallow conical shells and panels of uniform and non-uniform thickness [13–17].

On the other hand, dynamic buckling problems of uniform and non-uniform thin conical shells, under the effect of suddenly increasing pressure loads, have not been studied sufficiently. In particular, different theoretical solutions of these problems do not conform with the experimental results. This stems from the difficulty in theoretical solutions to consider all the factors (the variation of the loads with time, the scattering of waves in materials, etc.) affecting the behaviour of deformable systems under dynamic loads. Due to this fact, during recent years, the theoretical and experimental analyses of shells under different loading conditions and the determination of the pertinent functional relations concerning critical loads have attracted the attention of researchers. Sachenkov and Klementev [18] have studied the dynamic stability of elastic conical shells subjected to an external pressure, which is a linear function of time. Baktieva et al. [19] have studied the dynamic stability of cylindrical and conical shells under the same type of loading. In both studies it has been shown that the results found by using approximate formulas for critical loads, taking some coefficients therein from experimental results, are acceptable.

In practical applications, liquid and wind pressures are sometimes confronted as power functions of time, as well as linear and periodical ones. The solution of the dynamic buckling problems of shells under such loads comprises mainly of the determination of the dynamic factor. This factor, which is defined as the ratio of the dynamic critical load to the static one, can be found by different methods depending on the manner in which the load varies. For example, Shumik [20,21] has solved the buckling problems of cylindrical and conical shells subjected to a pressure, which is a power function of time, employing the energy and Lagrange methods applying Runge–Kutta numerical integration. Sofiyev and Aksogan [22,23] and Aksogan and Sofiyev [24,25] have solved the same problem, analytically, using a Ritz type variational method.

The effect of the variation of thickness on the dynamic buckling load of a conical shell has not been studied appreciably. The aim of the present study is to investigate the dynamic buckling of a truncated conical shell having a meridional thickness expressed by an arbitrary function subject to a uniform external pressure, which is a power function of time, using a Ritz type variational method.

2. Problem formulation

Fig. 1 shows a truncated conical shell. Let the co-ordinate system be chosen such that the origin O is at the vertex of the whole cone, on the middle surface of the shell, and s-axis lies on the curvilinear middle surface of the cone, s_1 and s_2 being the co-ordinates of the points where this axis intersects the small and large bases, respectively. The average radii of the small and large bases are r_1 and r_2, respectively, h is the plate thickness and γ is the semi-vertex angle. Furthermore, ζ-axis is always normal to the moving s-axis, lies in the plane of the s-axis and the axis of the cone and points inwards. θ is the angle of rotation around the longitudinal axis starting from a radial plane. Here h is a continuous and second order differentiable function of s.

For the shell described above the stress-strain relations are as follows:

$$\begin{pmatrix} \sigma_s \\ \sigma_\theta \\ \sigma_{s\theta} \end{pmatrix} = \frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & 1-v \end{bmatrix} \begin{pmatrix} \varepsilon_s \\ \varepsilon_\theta \\ \varepsilon_{s\theta} \end{pmatrix}, \tag{1}$$

where σ_s, σ_θ and $\sigma_{s\theta}$ are the stress components, $\varepsilon_s, \varepsilon_\theta$ and $\varepsilon_{s\theta}$ are the strain components, E is the elasticity modulus and v is the Poisson ratio of the isotropic material.

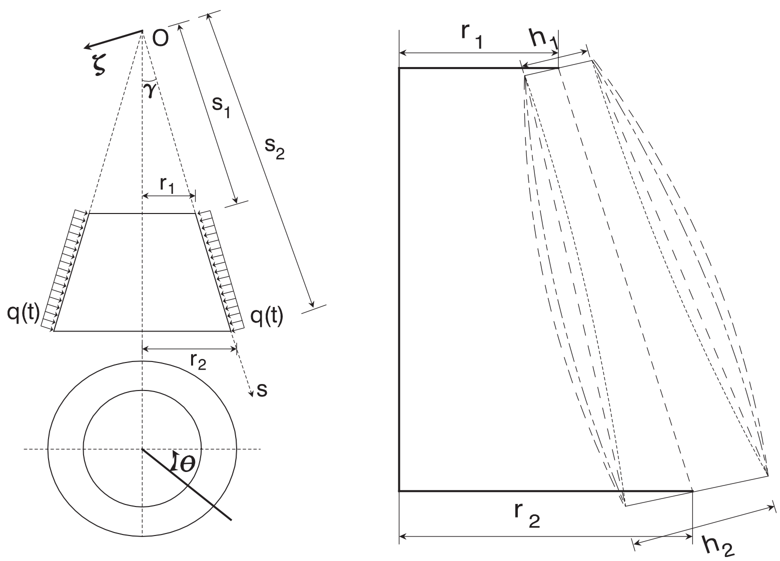

Fig. 1. Geometry of a truncated conical shell with variable thickness under a uniform external pressure. - - -, linear; – · · –, parabolic; – · –, cubic; · · · · · ·, exponential.

In Love's first approximation shell theory, the strain components are given by [26–30]

$$[\varepsilon_s, \varepsilon_\theta, \varepsilon_{s\theta}] = \left[e_s - \zeta\frac{\partial^2 u_3}{\partial s^2}, e_\theta - \zeta\left(\frac{1}{s^2}\frac{\partial^2 u_3}{\partial\phi^2} + \frac{1}{s}\frac{\partial u_3}{\partial s}\right), e_{s\theta} - \zeta\left(\frac{1}{s}\frac{\partial^2 u_3}{\partial s\partial\phi} - \frac{1}{s^2}\frac{\partial u_3}{\partial\phi}\right)\right], \tag{2}$$

where $\phi = \theta\sin\gamma$, e_s and e_θ are the normal strains in directions s and θ on the middle surface, respectively, $e_{s\theta}$ is the shear strain of the middle surface, and u_3 is the displacement of the middle surface in the normal direction, positive towards the axis of the cone.

The strain components on the middle surface of the shell are obtained employing the following approximate expressions [28–30]:

$$[e_s, e_\theta, e_{s_\theta}] = \left[\frac{\partial u_1}{\partial s}, \frac{1}{s}\frac{\partial u_2}{\partial\phi} + \frac{u_1}{s} - \frac{u_3\cot\gamma}{s}, \frac{1}{2}\left(\frac{1}{s}\frac{\partial u_1}{\partial\phi} + \frac{\partial u_1}{\partial s} - \frac{u_2}{s}\right)\right], \tag{2a}$$

in which u_1 and u_2 are, respectively, the displacements on the middle surface in the directions of s and θ.

The following integrals define the force and moment resultants [26–30]:

$$(n_s, n_\theta, n_{s\theta}) = \int_{-h/2}^{h/2}(\sigma_s, \sigma_\theta, \sigma_{s\theta})\,\mathrm{d}\zeta, \quad (m_s, m_\theta, m_{s\theta}) = \int_{-h/2}^{h/2}(\sigma_s, \sigma_\theta, \sigma_{s\theta})\zeta\,\mathrm{d}\zeta. \tag{3}$$

Introducing an Airy stress function F, the internal forces $(n_s, n_\theta, n_{s\theta})$ are given as follows:

$$(n_s, n_\theta, n_{s\theta}) = \left(\frac{1}{s^2}\frac{\partial^2 F}{\partial\phi^2} + \frac{1}{s}\frac{\partial F}{\partial s}, \frac{\partial^2 F}{\partial s^2}, -\frac{1}{s}\frac{\partial^2 F}{s\partial s\partial\phi} + \frac{1}{s^2}\frac{\partial F}{\partial\phi}\right). \tag{4}$$

According to the flexural shell theory, the dynamic stability and compatibility equations for truncated conical shells are given in the following form [27,29]:

$$\frac{\partial^2 m_s}{\partial s^2} + \frac{2}{s}\frac{\partial m_s}{\partial s} + \frac{2}{s}\frac{\partial^2 m_{s\theta}}{\partial s\partial\phi} - \frac{1}{s}\frac{\partial m_\theta}{\partial s} + \frac{2}{s^2}\frac{\partial m_{s\theta}}{\partial\phi} + \frac{1}{s^2}\frac{\partial^2 m_\theta}{\partial\phi^2} + \frac{n_\theta}{s}\cot\gamma$$
$$+ n_s\frac{\partial^2 u_3}{\partial s^2} + \frac{n_\theta}{s}\left(\frac{1}{s}\frac{\partial^2 u_3}{\partial\phi^2} + \frac{\partial u_3}{\partial s}\right) + 2n_{s\theta}\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial u_3}{\partial\phi}\right) - \rho h\frac{\partial^2 u_3}{\partial t^2} = 0, \tag{5}$$

$$\frac{\cot\gamma}{\mathrm{s}}\frac{\partial^2 u_3}{\partial s^2} - \frac{2}{s}\frac{\partial^2 e_{s\theta}}{\partial s\partial\phi} - \frac{2}{s^2}\frac{\partial e_{s\theta}}{\partial\phi} + \frac{\partial^2 e_\theta}{\partial s^2} + \frac{1}{s^2}\frac{\partial^2 e_s}{\partial\phi^2} + \frac{2}{s}\frac{\partial e_\theta}{\partial s} - \frac{1}{s}\frac{\partial e_s}{\partial s} = 0. \tag{6}$$

According to the membrane theory of shells, the dynamic buckling equation is given in the following form with loading terms in the coefficients of the derivatives of displacement u_3:

$$\frac{\partial^2 m_s}{\partial s^2} + \frac{2}{s}\frac{\partial m_s}{\partial s} + \frac{2}{s}\frac{\partial^2 m_{s\theta}}{\partial s\partial\phi} - \frac{1}{s}\frac{\partial m_\theta}{\partial s} + \frac{2}{s^2}\frac{\partial m_{s\theta}}{\partial\phi} + \frac{1}{s^2}\frac{\partial^2 m_\theta}{\partial\phi^2} + \frac{n_\theta}{s}\cot\gamma + (n_s^0 + \bar{n}_s)\frac{\partial^2 u_3}{\partial s^2}$$
$$+ \left(\frac{n_\theta^0}{s} + \frac{\bar{n}_\theta}{s}\right)\left(\frac{1}{s}\frac{\partial^2 u_3}{\partial\phi^2} + \frac{\partial u_3}{\partial s}\right) + 2(n_{s\theta}^0 + \bar{n}_{s\theta})\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial u_3}{\partial\phi}\right) - \rho h\frac{\partial^2 u_3}{\partial t^2} = 0, \tag{7}$$

where n_s^0, n_θ^0 and $n_{s\theta}^0$ are the incremental force resultants about the fundamental state. When the displacement u_3 is much less than the shell thickness, i.e., in the small displacement theory

$$\bar{n}_s\frac{\partial^2 u_3}{\partial s^2}, \quad \frac{\bar{n}_\theta}{s}\left(\frac{1}{s}\frac{\partial^2 u_3}{\partial\phi^2} + \frac{\partial u_3}{\partial s}\right), \quad 2\bar{n}_{s\theta}\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial u_3}{\partial\phi}\right)$$

terms being negligible, Eq. (7) simplifies to the following form [27–31]:

$$\frac{\partial^2 m_s}{\partial s^2} + \frac{2}{s}\frac{\partial m_s}{\partial s} + \frac{2}{s}\frac{\partial^2 m_{s\theta}}{\partial s \partial \phi} - \frac{1}{s}\frac{\partial m_\theta}{\partial s} + \frac{2}{s^2}\frac{\partial m_{s\theta}}{\partial \phi} + \frac{1}{s^2}\frac{\partial^2 m_\theta}{\partial \phi^2} + \frac{n_\theta}{s}\cot\gamma$$
$$+ n_s^0 \frac{\partial^2 u_3}{\partial s^2} + \frac{n_\theta^0}{s}\left(\frac{1}{s}\frac{\partial^2 u_3}{\partial \phi^2} + \frac{\partial u_3}{\partial s}\right) + 2n_{s\theta}^0 \frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial u_3}{\partial \phi}\right) - \rho h \frac{\partial^2 u_3}{\partial t^2} = 0. \tag{7a}$$

In cases without initial bending moment, Eq. (7a) simplifies to the following form:

$$\frac{n_\theta^0}{s}\cot\gamma + \rho h \frac{\partial^2 u_3}{\partial t^2} = -q_1 - q_0 t^n, \tag{7b}$$

where q_0 is the loading parameter, q_1 is the static external pressure, n is a positive whole number power satisfying $n \geqslant 1$ and expressing the time dependence of the external pressure. It has been proved by numerical computations that when the duration of external loading is greater than 10^{-3} s, disregarding the inertia term in Eq. (7b) does not affect the results. Due to this fact, when the shell is under an external pressure, which varies as a power function of time, Eq. (7a) yields the following expressions:

$$n_s^0 = 0, \quad n_\theta^0 = -s(q_1 + q_0 t^n)\tan\gamma, \quad n_{s\theta}^0 = 0. \tag{7c}$$

3. Solution of the problem

Employing Eqs. (1)–(4) and (7c) in Eqs. (6) and (7a), a pair of differential equations is obtained for u_3 and F, which can be written in matrix form as follows:

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{pmatrix} F \\ u_3 \end{pmatrix} = 0, \tag{8}$$

where the functional operators Q_{ij} $(i,j = 1,2)$ are defined in the following manner:

$$Q_{11} = \frac{12(1-\nu^2)}{E}\frac{\cot\gamma}{s}\frac{\partial^2}{\partial s^2}, \quad Q_{22} = \frac{E\cot\gamma}{s}\frac{\partial^2}{\partial s^2}, \tag{9a, b}$$

$$Q_{12} = \left[\frac{2(1-\nu)h^3}{s^4} + \frac{3-4\nu}{s}\frac{\partial}{\partial s}\left(\frac{h^3}{s^2}\right) - \nu\frac{\partial^2}{\partial s^2}\left(\frac{h^3}{s^2}\right)\right]\frac{\partial^2}{\partial \phi^2}$$
$$- 2\left[\frac{1}{s^2}\frac{\partial h^3}{\partial s} - \frac{h^3}{s^3}\right]\frac{\partial^3}{\partial s \partial \phi^2} - h^3\left[\frac{1}{s^4}\frac{\partial^4}{\partial \phi^4} + \frac{2}{s^2}\frac{\partial^4}{\partial s^2 \partial \phi^2} + \frac{\partial^4}{\partial s^4}\right]$$
$$- 2\left[\frac{\partial h^3}{\partial s} + \frac{h^3}{s}\right]\frac{\partial^3}{\partial s^3} - \left[\frac{\partial^2 h^3}{\partial s^2} + \frac{(2+\nu)}{s}\frac{\partial h^3}{\partial s} - \frac{h^3}{s^2}\right]\frac{\partial^2}{\partial s^2}$$
$$- \left[\nu\frac{\partial^2}{\partial s^2}\left(\frac{h^3}{s}\right) - \frac{1-2\nu}{s}\frac{\partial}{\partial s}\left(\frac{h^3}{s}\right)\right]\frac{\partial}{\partial s} - \frac{12(1-\nu^2)}{E}\left[\frac{q_1 + q_0 t^n}{\cot\gamma}\left(\frac{1}{s}\frac{\partial^2}{\partial \phi^2} + \frac{\partial}{\partial s}\right) + \rho h\frac{\partial^2}{\partial t^2}\right], \tag{9c}$$

$$Q_{21}=\frac{1}{h}\left[\frac{1}{s^4}\frac{\partial^4}{\partial\phi^4}+\frac{2}{s^2}\frac{\partial^4}{\partial\phi^2\partial s^2}+\frac{\partial^4}{\partial s^4}\right]$$
$$+2\left[\frac{1}{s^2}\frac{\partial h^{-1}}{\partial s}-\frac{1}{s^3h}\right]\frac{\partial^3}{\partial\phi^2\partial s}-\left[\frac{(3+4v)}{s}\frac{\partial}{\partial s}\left(\frac{1}{s^2h}\right)+\frac{2(1+v)}{s^4h}+v\frac{\partial^2}{\partial s^2}\left(\frac{1}{s^2h}\right)\right]\frac{\partial^2}{\partial\phi^2}$$
$$+2\left[\frac{\partial h^{-1}}{\partial s}+\frac{1}{sh}\right]\frac{\partial^3}{\partial s^3}+\left[\frac{\partial^2h^{-1}}{\partial s^2}-\frac{2v}{s}\frac{\partial h^{-1}}{\partial s}+\frac{2v}{s^2h}\right]\frac{\partial^2}{\partial s^2}-\left[v\frac{\partial^2}{\partial s^2}\left(\frac{1}{sh}\right)+\frac{(2v+1)}{s}\frac{\partial}{\partial s}\left(\frac{1}{sh}\right)\right]\frac{\partial}{\partial s}. \tag{9d}$$

Considering the shell to be simply supported along the peripheries of both bases, the solution of the system of Eq. (8) is sought in the following form:

$$u_3=\sum_i\sum_j\xi_{ij}(t)e^{\lambda\eta}\sin j_1\eta\cos i_1\phi,\quad F=\sum_i\sum_j\psi_{ij}(t)s_2e^{(\lambda+1)\eta}\sin j_1\eta\cos i_1\phi, \tag{10}$$

where $\xi_{ij}(t)$ and $\psi_{ij}(t)$ are time-dependent amplitudes, $j_1=j\pi/\eta_1, i_1=i/\sin\gamma$, $\eta=\ln(s/s_2)$, $\eta_1=\ln(s_2/s_1)$, i is the wave number in the θ direction and j is the half wave number in the s direction, which is equal to unity for the present case. For a truncated conical shell the parameter λ depends on the geometric parameter η_1 as follows [18]:

$$\eta_1<2.7\to\lambda=1.2,\quad 2.7\leqslant\eta_1\leqslant3.5\to\lambda=1.6,\quad \eta_1>2.7\to\lambda=2.0. \tag{11}$$

Differentiating Eq. (8) with respect to ϕ and s, each at a time, it is noted that, the functions involved in them should be steeply increasing with respect to ϕ and varying slowly with respect to s. Taking these properties into consideration, carrying out the transformation $\eta=\ln(s/s_2)$, neglecting small terms, multiplying the first equation by $u_3s_2^2e^{2\eta}\,d\eta\,d\phi$ and the second by $Fs_2^2e^{2\eta}\,d\eta\,d\phi$, considering Eq. (10) for $0\leqslant\phi\leqslant2\pi\sin\gamma$ and $-\eta_1\leqslant\eta\leqslant0$, applying Galerkin's method and taking into consideration from the series the terms $\xi_{ij}(t)$ and $\psi_{ij}(t)$ and eliminating the latter, the following equation is obtained:

$$\frac{d^2\xi_{ij}(\tau)}{d\tau^2}+(\lambda_1-\lambda_2\tau^n)\xi_{ij}(\tau)=0, \tag{12}$$

in which the following definitions apply:

$$\lambda_1=\frac{t_{cr}^2(B_1i_1^4+B_2i_1^{-4}-q_1B_3i_1^2)}{\rho},\quad \lambda_2=\frac{q_0B_3i_1^2t_{cr}^{n+2}}{\rho}$$

$$B_1=\frac{E}{12(1-v^2)}\frac{A_1}{s_2^4A_4},\quad B_2=\frac{EA_2A_6\cot^2\gamma}{s_2^2A_4A_5},\quad B_3=\frac{\tan\gamma}{s_2}\frac{A_3}{A_4},$$

$$A_1=\int_{-\eta_1}^0h^3e^{2(\lambda-1)\eta}\sin^2j_1\eta\,d\eta,\quad A_2=\int_{-\eta_1}^0\left[\frac{\partial^2(e^{(\lambda+1)\eta}\sin j_1\eta)}{\partial\eta^2}\right]e^{(\lambda-1)\eta}\sin j_1\eta\,d\eta, \tag{13}$$

$$A_3=\int_{-\eta_1}^0e^{(2\lambda+1)\eta}\sin^2j_1\eta\,d\eta,\quad A_4=\int_{-\eta_1}^0h\,e^{2(\lambda+1)\eta}\sin^2j_1\eta\,d\eta,$$

$$A_5=\int_{-\eta_1}^0\frac{e^{2\lambda\eta}\sin^2j_1\eta}{h}d\eta,\quad A_6=\int_{-\eta_1}^0\left[\frac{\partial^2(e^{\lambda\eta}\sin j_1\eta)}{\partial\eta^2}\right]e^{\lambda\eta}\sin j_1\eta\,d\eta,$$

where $t = t_{cr}\tau$, t_{cr} is the critical time and τ is the dimensionless time parameter varying in the interval $0 \leqslant \tau \leqslant 1$. Approximating function will be chosen as

$$\xi_{ij}(\tau) = A_{ij}\xi(\tau) = A_{ij}\tau\left[(m+2)(m+1)^{-1} - \tau\right]\exp(m\tau) \tag{14}$$

and satisfies the initial conditions $\xi(0) = 0$, $\partial\xi(1)/\partial\tau = 0$. Here, m is an unknown coefficient and amplitude A_{ij} is found from the condition of transition to static condition.

Multiplying Eq. (12) by $\xi'_{ij}(\tau)$ and integrating it with respect to τ, from 0 to τ and from 0 to 1, in that order, the Ritz type variational method yields the following characteristic equation for finding the critical load [19,23]:

$$q_0 t_{cr}^n = C_1\left[\frac{B_1}{B_3} i_1^2 + \frac{B_2}{B_3}\frac{1}{i_1^6} - q_1\right] + \frac{C_2\rho}{t_{cr}^2 B_3}\frac{1}{i_1^2}, \tag{15}$$

in which the new constants are defined as follows:

$$C_1 = \frac{\int_0^1[\xi(\tau)]^2\,\mathrm{d}\tau}{2\int_0^1\int_0^\tau \eta^n\xi'(\eta)\xi(\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau}, \quad C_2 = \frac{\int_0^1[\xi'(\tau)]^2\,\mathrm{d}\tau}{2\int_0^1\int_0^\tau \eta^n\xi'(\eta)\xi(\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau}. \tag{16}$$

If expression (15) is minimized with respect to the parameter i_1^2, after some operations the following equation is found for finding the minimum critical load:

$$q_0 t_{cr}^n = 2C_1\left[\frac{B_1}{B_3} i_1^2 - \frac{B_2}{B_3}\frac{1}{i_1^6} - \frac{q_1}{2}\right]. \tag{17}$$

For the static condition ($t_{cr} \to \infty$, $q_0 \to 0$) the wave number is found as

$$i_{st}^8 = 3B_2B_1^{-1}. \tag{18}$$

For $q_1 = 0$, substituting Eq. (18) into Eq. (17) and replacing $q_0 t_{cr}^n / C_1$ by q_{cr}^{st}, the static critical load is found as

$$q_{cr}^{st} = 1.7548 B_1^{3/4} B_2^{1/4} B_3^{-1}. \tag{19}$$

For large values of q_0, eliminating t_{cr} from Eqs. (15) and (17), solving the resulting equation for the wave parameter i_1 and taking the relation $i_1 = i/\sin\gamma$ into consideration, one finds

$$i_d^2 = B_1^{-1/4} B_2^{1/4} B_4^{n/(2+2n)}, \tag{20}$$

where i_d is the wave number corresponding to the dynamic critical load and the following definition applies:

$$B_4 = \frac{C_2 B_3^{2/n} q_0^{2/n}\rho}{2^{2/n} C_1^{(n+2)/n} B_1^{0.5(n+3)/n} B_2^{0.5(n+1)/n}}. \tag{21}$$

The dynamic critical load is found, putting $q_1 = 0$ and substituting Eq. (20) into Eq. (17), as follows:

$$q_{cr}^d = q_0 t_{cr}^n = 2C_1 B_1^{3/4} B_2^{1/4} B_3^{-1} B_4^{n/(2n+2)}. \tag{22}$$

The value of coefficient m for which this dynamic critical load takes its minimum value is found as the ordinate of the minimum point of (q_{cr}^d, m) parabola and it can be shown by numerical computations that, for an external pressure given as a power function of time, $m = n + 1$.

From the definition $K_d = q_{cr}^d / q_{cr}^{st}$, the dynamic factor is found as

$$K_d = 1.1398 B_4^{n/(2+2n)}. \tag{23}$$

The critical stress impulse is obtained as

$$I_{cr} = \int_0^{t_{cr}} q_0 t^n \, \mathrm{d}t = q_0 t_{cr}^{n+1}/(n+1) = \bar{B}_4^{1/2} [2 C_1 B_1^{3/4} B_2^{1/4} B_3^{-1}]^{(n+1)/n}/(n+1), \tag{24}$$

where

$$\bar{B}_4 = \frac{C_2 B_3^{2/n} \rho}{2^{2/n} C_1^{(n+2)/n} B_1^{0.5(n+3)/n} B_2^{0.5(n+1)/n}}. \tag{25}$$

The appropriate formulae for a uniform truncated elastic conical shell are found as a special case, by putting $h=$ const. in expressions (18)–(24).

4. Numerical computations and results

The case of a conical shell, in which the thickness changes as a power or an exponential function in the axial direction, has been considered by Irie et al. [2]. Consider a conical shell whose thickness is expressed as (see Fig. 1)

$$h = h_2 - (h_2 - h_1)[(1 - \mathrm{e}^{\eta})/(1 - \mathrm{e}^{-\eta_1})]^k, \quad k > 0, \tag{26}$$

$$h = h_2 (h_1/h_2)^{(1-\mathrm{e}^{\eta})/(1-\mathrm{e}^{-\eta_1})}, \tag{27}$$

where $k = 1, 2, 3$ correspond to the cases of linear, parabolic and cubic variations, respectively.

Computations have been carried out for the case where the material properties are $E = 7.75 \times 10^4\,\mathrm{MPa}$, $v = 0.3$, $\rho = 3.1 \times 10^3\,\mathrm{kg/m^3}$, the shell parameters are $h_2/r_2 = 0.01$, $r_1 = 3 \times 10^{-2}\,\mathrm{m}$, $r_2 = 0.3\,\mathrm{m}$ and the loading parameter is $q_0 = 500\,\mathrm{MPa/s}^n$ [18,20] by considering different values of n and the results, thus obtained, have been presented in the form of graphs and a table.

Fig. 2 shows the values of the dynamic critical load and dynamic factor, for a non-uniform conical shell, versus the thickness ratio h_1/h_2. In general, the values of the dynamic critical load increase with an increase in the ratio h_1/h_2, whereas the values of the dynamic factor decrease. When $h_1/h_2 = 1$, the values of the dynamic critical load and dynamic factor for the uniform shell are obtained. The differences among the values of the dynamic critical loads (or the dynamic factors) for shells with linearly, parabolically, cubically or exponentially varying thickness, increase, with a decrease in the ratio h_1/h_2. Even small changes in the ratio h_1/h_2 affect the values of the critical parameters to a relevant extent. As examples, for the ratio $h_1/h_2 = 0.75$, the percentage differences in the values of q_{cr}^d and K_d, compared to the values for a shell of uniform thickness, are, respectively: 1-Exponential variation: %16.51 and %35.8, 2-linear variation: %15.83 and %34.22, 3-parabolic variation: %11 and %23.63, 4-cubic variation: %8.12 and %17.33. It can be seen that the effect of thickness variation on the dynamic factor is more than that on the critical load. Furthermore, with a decrease in the ratio h_1/h_2, the effect on the critical parameters gets even greater. The effects on the dynamic critical load and dynamic factor are almost same, when the thickness ratio is in the range $0.95 \leqslant h_1/h_2 \leqslant 1$.

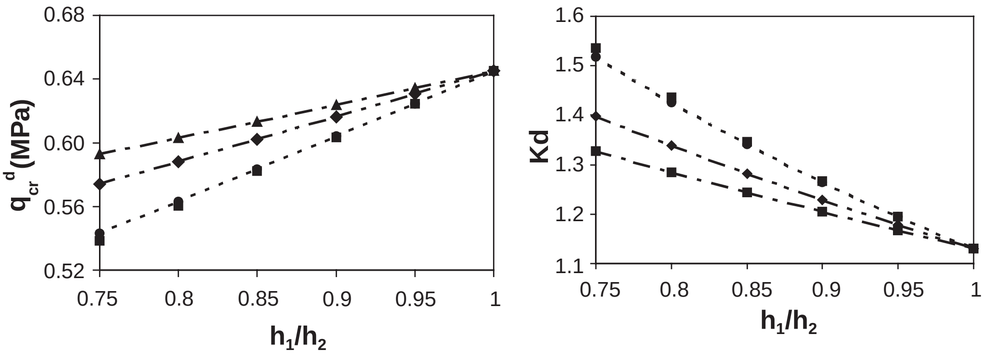

Fig. 2. Variation of the dynamic critical load and dynamic factor with variable thickness. - - -, linear; — · · —, parabolic; — · —, cubic; · · · · · ·, exponential ($\gamma = 70°$, $n = 1$, $m = 2$).

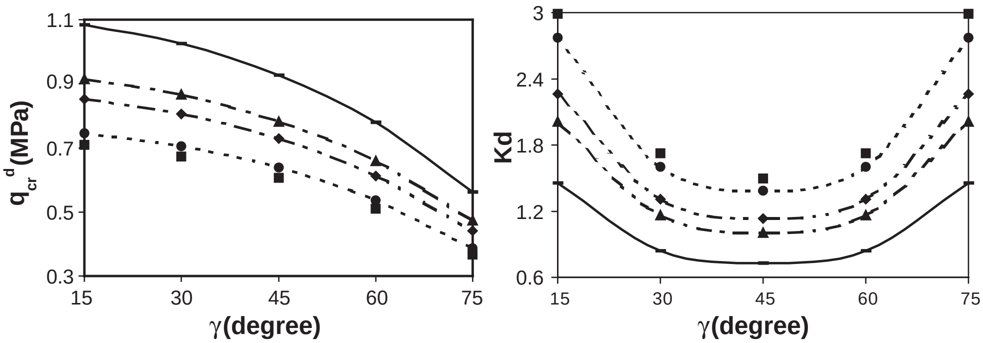

Fig. 3. Variation of the dynamic critical load and dynamic factor with γ. - - -, linear; — · · —, parabolic; — · —, cubic; · · · · · ·, exponential; ——, uniform ($h_1/h_2 = 0.5$, $n = 1$, $m = 2$).

Fig. 3 shows the values of the dynamic critical load and dynamic factor, for different thickness variations versus the values of the semi-vertex angle γ. The values of the dynamic critical load decrease with an increase in the semi-vertex angle γ for shells with uniform and varying thickness. It is observed that the effect of thickness variation on the values of the critical parameters does not change with γ, appreciably. Furthermore, for varying values of the ratio h_1/h_2 the curves in Fig. 3 shift with nearly the same form parallel to the vertical axis, but the dynamic factor takes its minimum value for $\gamma = 45°$ for all cases.

Fig. 4 shows the values of the dynamic critical load and dynamic factor versus the time-dependent pressure variation. When n increases, the dynamic critical load and dynamic factor decrease. Comparing the uniform shells with the shells of varying thickness, the importance of the difference between the values of critical parameters can be seen. If $n \geqslant 3$, the dynamic critical load (or the dynamic factor) is almost unchanged when the shell thickness varies as a power or an exponential function.

The values of the critical stress impulse corresponding to various values of the ratio h_1/h_2 are presented in Table 1 for different loads. With a decrease in the ratio h_1/h_2, the effect of thickness change on the critical stress impulse values increases appreciably. For example, for $h_1/h_2 = 0.6$,

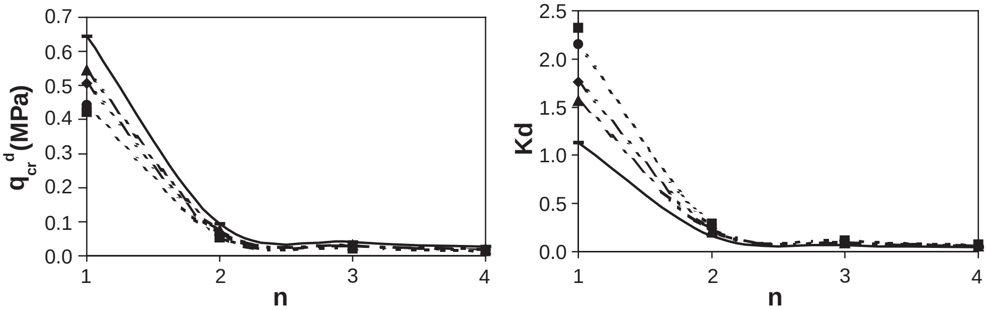

Fig. 4. Variation of the dynamic critical load and dynamic factor with n. ---, linear; $-\cdot\cdot-$, parabolic; $-\cdot-$, cubic; $\cdot\cdot\cdot\cdot\cdot\cdot$, exponential; ——, uniform ($h_1/h_2 = 0.5$, $\gamma = 70°$, $m = n + 1$).

Table 1
Variation of the critical stress impulse values with q_0 and h_1/h_2 ($\gamma = 70°$, $n = 1$, $m = 2$)

$I_{cr} \times 10^4$ (MPa s)				
h_1/h_2	Exponential	Linear	Parabolic	Cubic
0.6	2.211	2.331	2.843	3.169
0.8	3.141	3.172	3.458	3.634
1.0	4.161	4.161	4.161	4.161

compared to a shell of uniform thickness, shells of thickness varying exponentially, linearly, parabolically and cubically have critical stress impulses differing by %46.86, %43.98, %31.68 and %23.84, respectively. Another point to be noted is that the critical stress impulse depends on the loading parameter (see Eq. (24)).

The same problem for uniform truncated conical shell was solved numerically, using energy method, Lagrange equation and Runge–Kutta method by Shumik [21]. Sachenkov and Klementev [18] solved the same problem using experimental method. The comparisons with those methods were carried out for the following material properties and shell and loading parameters [18,21]: $E = 2.11 \times 10^5$ MPa, $v = 0.3$, $\rho = 8 \times 10^3\,\mathrm{kg/m^3}$, $h = 1.3 \times 10^{-4}$ m, $r_2 = 8 \times 10^{-2}$ m, $r_1 = 2.25 \times 10^{-2}$ m, $n = 1$, $m = 2, q_0 = 225$ MPa/s, $\gamma = 30°$.

The present method gives the critical dynamic load as $q^d_{cr} = 0.0764$ MPa and the dynamic factor as $K_d = 2.9882$, which are not too much different from those given as experimental results $q^d_{cr} = 0.0726$ MPa, $K_d = 2.69$ [18] and those given as numerical results $q^d_{cr} = 0.0736$ MPa, $K_d = 2.8790$ [21].

5. Conclusions

An investigation has been carried out on the buckling of truncated conical shells with variable thickness, subject to a uniform external pressure, which is a power function of time. The analytical solutions for the dynamic and static critical loads and the corresponding wave numbers, the dynamic factor and the critical stress impulse for conical shells with varying thickness have been found. Numerical computations were carried out for power and exponential variations of the thickness of the shell, the variation of the semi-vertex angle and the variation of the power of time in the external pressure expression. The results obtained were presented in the form of graphs and a table. The fair match of the values of the dynamic critical load and dynamic factor for a uniform conical shell with the experimental and numerical ones given in literature [18,21] has supported the validity of the present method.

Appendix A. Nomenclature

E	elasticity modulus of the isotropic material
$e_s, e_\theta, e_{s\theta}$	strain components on the middle surface of the conical shell
F	stress function
h	thickness of the conical shell
h_1, h_2	thickness of the conical shell at the small and large bases
i	wave number in the circumferential direction
i_{st}, i_d	wave number corresponding to the static and dynamic critical loads
I_{cr}	critical stress impulse
j	half wave number in the s direction
K_d	dynamic factor
Q_{ij}	functional operator
$m_s, m_\theta, m_{s\theta}$	moment resultants
$n_s, n_\theta, n_{s\theta}$	force resultants
$n_s^0, n_\theta^0, n_{s\theta}^0$	incremental force resultants about the fundamental state
n	power of time in the external pressure expression
q_{cr}^{st}, q_{cr}^{d}	static and dynamic critical loads
q_0, q_1	loading parameter and static external pressure
r_1, r_2	average radii of the small and large bases of the conical shell
s	the co-ordinate axis through the vertex on the curvilinear middle surface
s_1, s_2	the inclined distances of the bases of the cone from the vertex
t, t_{cr}	time and critical time
u_1, u_2, u_3	displacements on the middle surface in the directions of s, θ and ζ axes
$\varepsilon_s, \varepsilon_\theta, \varepsilon_{s\theta}$	strain components
γ	the semi-vertex angle of the cone
θ	the angular co-ordinate around the longitudinal axis from a radial plane
τ	dimensionless time parameter
λ	parameter that depends on the geometry of the shell
ρ	density of the elastic isotropic material
v	the Poisson ratio of the isotropic material
$\sigma_s, \sigma_\theta, \sigma_{s\theta}$	stress components
$\xi(t), \psi(t)$	time-dependent amplitudes
ζ	the co-ordinate axis in the inwards normal direction of the middle surface

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